Classification of surfaces. Knots revisited

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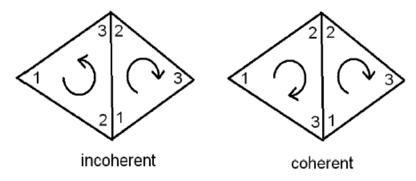
Orientable triangulations

Definition

A triangulation of a surface Σ is **orientable** if all faces can be oriented in a **coherent** way:



Orientability and Attaching lemma

Theorem

The following are equivalent:

1. *surface Σ has an orientable triangulation;*
2. **any** *triangulation of Σ is orientable;*
3. **any** *2-cell decomposition is orientable.*

If these properties hold, we say that the surface Σ is **orientable***.*

Lemma

Sphere with one handle and one Möbius band is homeomorphic to a sphere with 3 Möbius bands.

Compact surfaces

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Give example of a bounded surface which is not closed.

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Recall: a surface (in $\mathbb{R}^n$) is a geometric figure that is locally homeomorphic to $\mathbb{R}^2$.

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Surfaces are surfaces with (empty) boundary.

Classification theorem

Theorem

Any compact surface Σ is determined uniquely up to homeomorphism by the following data:

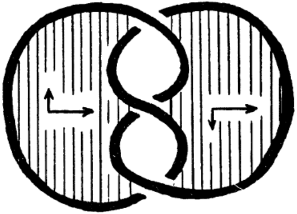
$$\chi(\Sigma),\ \textit{orientability}\ ,\ \textit{number of boundary components}$$

In particular, orientable surfaces with no boundary are just spheres with handles. Non-orientable surfaces with no boundary are just spheres with Möbius bands (at least one).

Definition

A **spanning surface** or **span** of a knot K is a surface in $\mathbb{R}^3$ (with no self intersections!) whose boundary is exactly the knot K.

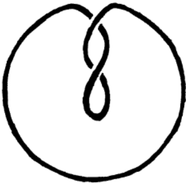
(a)



(b)

Exercise: unknot can be spanned by a Möbius band.

Another example



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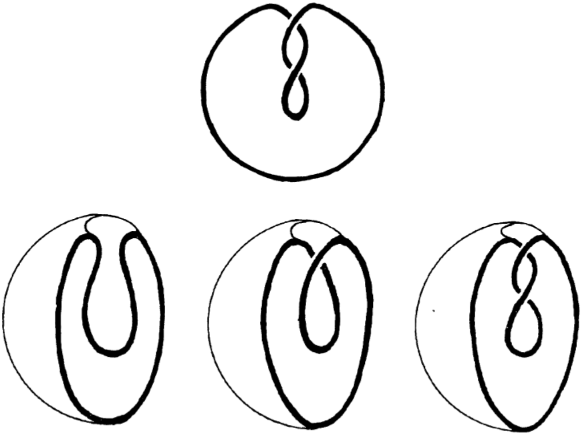

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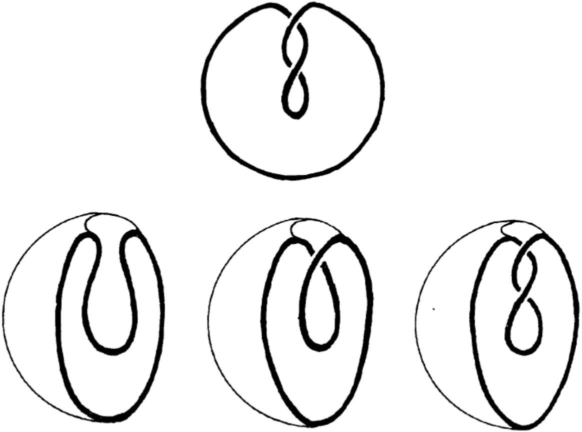

Exercise: Find an oriented span of figure 8 knot.

Big theorem

Theorem (Frankl-Pontryagin'40, Seifert'44)

Any knot or link can be spanned by an <u>oriented</u> surface.

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Walk on the knot, and jump at each crossing to another branch, always moving in the direction of the orientation of the knot.

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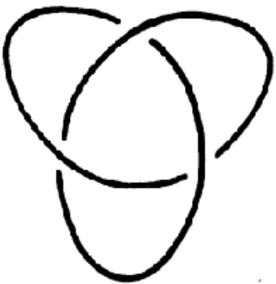
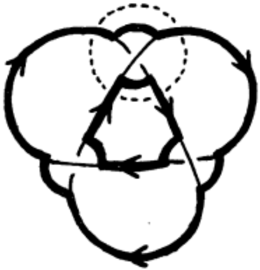
Keep doing that, until the whole knot is broken into closed curves.

Seifert cycles





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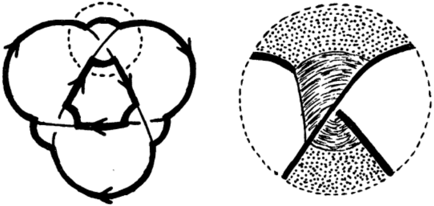

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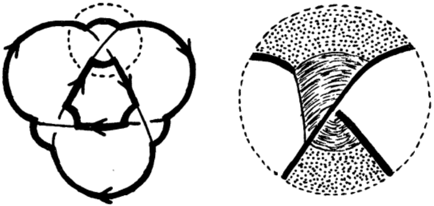

The last question is: why is this thing orientable?

End of Seifert algorithm

Orientations of polygons come from orientation on the knot.